Convex Hulls Surface Mapping onto a Sphere

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Figure 1: Iterative approach for dealing with coplanar triangles - (a) Calculate the shape centroid, (b) determine outer surface points using support mapping (i.e., the normal from the centroid to each point), (c) project surface points onto a unit sphere (i.e., centroid to surface point fixed length), (d) use the projected surface points to find the interconnected convex hull triangles, and (e) the indexes for each of the projected surface points are the convex hull.

Abstract

Writing an uncomplicated, robust, and scalable three-dimensional convex hull algorithm is challenging and problematic. This includes, coplanar and collinear issues, numerical accuracy, performance, and complexity trade-offs. While there are a number of methods available for finding the convex hull based on geometric calculations, such as, the distance between points, but do not address the technical challenges when implementing a usable solution (e.g., numerical issues and degenerate cloud points). We explain some common algorithm pitfalls and engineering modifications to overcome and solve these limitations. We present a novel iterative method using support mapping and surface projection to create an uncomplicated and robust 2d and 3d convex hull algorithm.

1 Introduction

Problem Convex hull algorithms are an essential multi-discipline technique important to several fields, such as, computer graphics, pattern recognition, medical analysis and design automation [Barber et al. 1996; Bentley and Shamos 1978; Gregorius 2014; Graham and Frances Yao 1983]. While multiple approaches are available (e.g., gift wrapping [Chand and Kapur 1970] and divideand-conquer [Preparata and Hong 1977]), writing a stable and robust three dimensional implementation is difficult and challenging. Since implementing an algorithm in two dimensions may be easy, but not so for three dimensions [Avis and Bremner 1995]. We survey a number of techniques and address common problems with convex hull algorithms in practice. Since highly complex and degenerate vertices arise in practice making it difficult to generate a reliable convex hull painlessly. We present a novel method of projecting the convex surface points onto a spherical boundary to remedy numerical sensitivity that produces a simple and reliable solution for both two and three dimensional problems.

Motivation The mathematics for generating a convex hull from a set of points is well defined, yet there is no de factor standard algorithm or implementation. A number of innovative and interesting concepts have been published that solve the problem, yet the implementation of a robust 3D convex hull algorithm is paved with technical challenges [Avis and Bremner 1995]. The emphasis of

this article is on a novel straightforward algorithm to produce a stable implementation that can be applied in both 2D and 3D easily.

Challenges While a number of innovative and original convex hull algorithms have been presented, they do not address practical short-comings. Typically, the algorithms are presented in the context of simple test cases, such as, two-dimensions, to explain the working concept. We focus on the engineering enhancement necessary for a real-world implementation. We create a variety of test cases to evaluate the success and failure of the implementation based upon specific problems (see Figure 3). Our solution does not aim for an optimal answer, instead we focus on an uncomplicated method that will generate an accurate convex hull reliably.

Previously As we would expect, the convex hull problem has been well studied over the past few decades and resulted in a number of solutions (see Figure 2). There are two main class of algorithm for solving convex hull problems: insertion algorithms and pivoting algorithms [Avis and Bremner 1995]. With our implementation sitting within the insertion regime. An example of a successful insertion method, is the ghull [Barber et al. 1996; Barber 1995-2012] algorithm, which solves precision issues caused by coplanar points by merging facets. This includes, merging a point into a coplanar facet, merging concave facets, merging duplicate ridges, and merging flipped facets. Similarly, a pivoting algorithm implementation used by the open source Bullet Physics Engine [Coumans 2012], is able to generate reliable convex hulls for collision detection problems. The Bullet convex hull implementation is based on Preparata and Hong [Preparata and Hong 1977] method. This has a time complexity of O(nlogn). Furthermore, to make the algorithm less sensitive to rounding errors, all computations are done with integer math. The algorithm handles degenerate cases, including arbitrary flat and parallel faces. While our method is based upon [Eddy 1977] (QuickHull), but reduces the computation of extremely sensitive collinear and coplanar issues by identifying outer hull points and projecting them onto a common spherical surface.

Contribution Our main contributions include a definition for a novel convex hull algorithm with less numerical sensitivity and the ability to deal with collinear and coplanar vertices. The algorithm presented in this article offers a number of desirable benefits:

- ✓ Easy to implement (i.e., both 2D and 3D)
- \checkmark Terminating condition

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Brute Force [A] O(n ⁴)	Graham Sc [C] O(n log 1972	an Q g n) [E	uick Hull E] O(nh) 1977-78	Monotone Chain [F] O(n log n) 1979	Marriage- [H] O(n lo	Marriage-before-Conquest [H] O(n log h) 1986	
·	1970	1973	. 1977		1984	1986	
Gift Wrapping [B] O(nh)		i Jarvis March [D] O(nh)	Divide [E] O(r	-and-Conquer 1 log n)	Incremental [G] O(n log n)	Delaunay Triangulatio	on and Convex Hull

Figure 2: *Timeline* - *Let n is the number of points in the input set, and h is the number of vertices on the output hull.* [*A*] *Brute Force* [*B*] [*Chand and Kapur* 1970] [*C*] [*Graham* 1972] [*D*] [*Jarvis* 1973] [*E*] [*Eddy* 1977; *Bykat* 1978] [*F*] [*Preparata and Hong* 1977] [*G*] [*Andrew* 1979] [*H*] [*Kallay* 1984] [*I*] [*Kirkpatrick and Seidel* 1986] [*J*] EPA - Expanding Polytope Algorithm [*Kim et al.* 2002] [*K*] [*Maus* 1984]

- ✓ Can be used with a tolerance in the expansion step for automatic simplification
- ✓ Handle 'co-planar' and 'degenerate' input data
- \checkmark The output mesh is built entirely from the input vertices
- ✓ Can be applied to real-world complex models, not just point clouds

Our approach is the engineering enhancement of mapping the surface points onto a uniform sphere to solve a number of technical shortcomings (e.g., user intervention (tweaking) and numerical issues common with coplanar and collinear faces).

2 Related Work

The convex hull problem has received considerable attention in computational geometry [Gregorius 2014; Graham and Frances Yao 1983; Kim et al. 2002; Maus 1984]. Computing a convex hull (or just a "hull") is one of the first sophisticated geometry algorithms, and there are many variations of it. The most common form of this algorithm involves determining the smallest convex set (called the "convex hull") containing a discrete set of points. This algorithm also applies to a polygon, or just any set of line segments, whose hull is the same as the hull of its vertex point set. There are numerous applications for convex hulls, for instance, collision avoidance, hidden object determination, and shape analysis.

The most popular hull algorithms are the "Graham scan" algorithm [Graham 1972] and the "divide-and-conquer" algorithm [Preparata and Hong 1977]. Implementations of both these algorithms are readily available (see [O'Rourke 1998]). Both are O(nlogn) time algorithms, but the Graham has a low run-time computation overhead in 2D. However, the Graham algorithm does not generalize to 3D and higher dimensions, whereas the divide-and-conquer algorithm has a natural extension. Figure 2 shows the time-line of hull algorithms.

Our work is based around the insertion algorithm concept. Where an initial convex hull approximation is created (i.e., a starting tetrahedron for 3D). We use the support mapping (e.g., see Expanding Polytope Algorithm (EPA) [Kim et al. 2002]) and Quickhull methodology to iteratively grow and encapsulate all the points to form a convex hull.

Which method is best? It depends on what you want, for example, do you want a 2D or a 3D solution? Are you concerned with run-time speeds or numerical stability and accuracy?

- Parallelizable
- Memory Overhead
- Complexity
- Computational Speed
- Robustness and Numerical Sensitivity
- Number of Dimensions (2D or 3D)

3 Background

Convex Hulls A convex hull means the smallest convex region which encloses a specified group of points. Technically, it is the smallest convex set containing the points, and can be visualized as a rubber band which wraps around the 'outside' points (i.e., all other points must lie within this rubber band [Barber et al. 1996]). A convex hull is different for dissimilar objects because it depends upon the feature point of every object. For a detailed explanation of Convex Geometry, see Joseph O'Rourke [O'Rourke 1998]. A convex hull of a set is unique (upto co-linearities). Our method of surface mapping and projecting the points onto a sphere reduces sensitivity and co-linearity ambiguities.

Support Mapping Support mapping is often used in physics and collision detection [Kim et al. 2002]. The support mapping for a cloud of points given a direction is the point that is farthest in the direction - which simply means finding the point with the maximum dot product (i.e., dot(direction, point). The supporting point in any direction is guaranteed to be on the surface of the convex hull cloud of points. We exploit this concept in our algorithm to efficiently determine the surface points.

No New Vertices We have an array of points and want to find how they can be connected using triangles to form a convex hull. No extra points are added. The vertices are numbered from 1 to N, with each triangle formed by an array of three indices into the vertex array. This is to avoid any numerical drifting. We work with triangle and vertex indices and do not generate any new points.



Figure 4: *Complex 3D Model* - *Complex model of a plant (19370 faces) mesh and generated convex shell (198 faces).*

Centroid A bounded convex polyhedron is called a polytope. The centroid of a convex polytope as the centroid of its vertices



Figure 3: Implementation Considerations - Example problems, such as, (a) multiple overlapping points, (b) co-planar points, (c) valid triangle, (d) degenerate triangle (i.e., long and thin causes issues when calculating face normals - sliver-shaped triangles), and (e) coplanar points - causing incorrect selection of faces for an expanding convex polytope (i.e., overlapping triangles resulting in concave errors as shown in Figure 9).

is given by Equation 1.

$$\frac{1}{n}\sum_{i=1}^{n}p_{i}\tag{1}$$

where the centroid is composed of a set of points $p_1, ..., p_n$. Note, the centroid will be contained within the relative interior of the convex hull.

Algorithm Overview Our algorithm adopts the well-known QuickHull approach but with additional pre-phase culling and remapping of the vertices. It starts by calculating the centroid and performing a support mapping phase to strip inner vertices, the final vertices are then mapped onto a spherical surface using the centroid. From the set of points we use four points to generate a tetrahedron (note - due to support mapping phase, we will not need to discards any internal points as the hull grows). It then iteratively refines the faces of the polyhedron by adding external points, and redistributes the remaining points associated with each face among its children faces. The refinement of a face is performed by selecting the furthest point from its associated points and generating three children triangles. We do not need to worry about concave edge swapping or removing concave vertices.

4 Our Iterative Method

The convex hull algorithm presented in this paper focuses on 3D cases. We are mainly interested in computing convex hulls that are able to solve unforeseen problems for arbitrary clouds of points, which can contain degenerate data, in addition to model scene geometry, such as, complex 3D geometric models (e.g., see Figure 6, Figure 5, and Figure 11).

A preliminary stage is run to strip out and prepair the points. The algorithm starts with a set of points. We remove duplicate points (i.e., points within a predefined tolerance). We calculate the centroid of the set of points). We calculate the normal from the centroid to each point and find the support vertex, and add it to a list. The list will contain a set of points which sit on the convex hull surface. Using the centroid, we project each surface point onto a sphere (i.e., with the centroid the centre of the sphere).

The iterative stage starts by taking the first four vertices and connecting them to form the smallest possible starting closed mesh (i.e., a tetrahedron). After we have set-up the tetrahedron, we iteratively grow the convex hull to encapsulate the rest of the surface points and create the convex hull. We grow the convex hull by going through each of the surface points, and selecting the triangle surfaces that are visible to the point. The visible faces are removed and a new set of faces are added using the edges are are not shared and the new surface point. The key stages are given in Algorithm 1.

- **Point inside or outside a convex shape** we can easily determine if a point is inside a convex hull by iterating over all the faces and checking if the point is on the inside of the plane (i.e., dot product). This can be useful for automatically checking if the algorithm failed when developing the implementation.
- Faces that a point can see we find all the faces that are visible to a point by taking the dot product of the face normal and the point (i.e., front facing if $n \bullet p > n \bullet v$, where *n* is the face normal, *v* is a face vertex, and *p* is the test point).
- Extract edges from a set of faces we have an array of edges from all the found triangles, any edges that are shared (i.e., *count* > 1), are thrown away. Then the remaining edges are used to create new triangles (i.e., edge and the new point).

Dynamics Our method is able to handle unknown sets of points. In addition, due our algorithms iterative nature, we are able to handle changing concave hulls, where points can be added or deleted on-the-fly. Our convex hull algorithm is easily able to update the mesh after each insertion/deletion operation.

Modifications The algorithm is flexible and can be modified to approach the problem in different ways. For example, instead of iteratively selecting each vertex in the list as we do in our implementation, we could exploit the support mapping concept further by iterating over each face and select the point furthest from the face to iteratively grow the convex polytope. Our algorithm relies on a simple local geometric point-plane test to determine the position of a point with respect to a plane, which is used to pick the triangles to merge the point with. As we have already culled inner points with the support mapping phase. All the points after the support mapping phase are used to create the convex hull surface. Projecting the points onto a sphere reduces co-planar and co-linear issues. Due to the approach the method does not require any swap operations to resolve fold-overs and self-intersections which can complicate the point-plane test and disturb their locality [Stein et al. 2012].

Optimisation We did not focus on any optimisations, but on a novel solution for providing a robust and easy to implement method that resolves common issues (i.e., reducing coplanar and collinear points). However, the timing results for our implementation are given in Figure 7 to shows the relationship between the number of vertices and the elapsed time. Our iterative algorithm has a time complexity $O(n \log n)$ since it is built upon the concept presented by Clarkson and Shor [Clarkson and Shor 1989] which iteratively

adds external point to extend the convex polyhedron until the remaining set of points becomes empty.

Algorithm 1: Iterative 3D Surface Mapping onto a Sphere Convex Hull Algorithm

Data: Array Points

- Result: Triangle array representing the convex hull for the set of Points
- Remove duplicate points (i.e., within pre-defined distance tolerance)
- Calculate Centroid 2
- 3 Empty array spherepoints
- 4 for *i*=1 to num points do
- Normal = Centroid to Points(i) 5
- supportpoint = FindSupportPoint(Normal, Points) 6
- Project surfacepoint onto sphere surface 7
- 8 Add supportpoint to spherepoints
- end 9

5

- 10 Array hull
- Construct a tetrahedron using spherepoints (e.g., first four points 11 points) add to array hull
- for *i*=4 to num spherepoints do 12
- Find all triangles that are visible to spherepoint(i) (i.e., front 13 of the triangle)
- Remove found triangles from array hull 14
- Find non-shared edges for the removed triangles 15
- Add new triangles using spherepoint(i) and non-shared edges 16 end 17

Experimental Results

We implemented the algorithm using floating point precision in C++ within Visual Studio 2013 and Windows-7. We evaluated our implementation using various test scenarios:

- Various complex 3D models (e.g., gun, rabbit, teapot) Figure 11,6
- Procedural test case (e.g., Minkowski shape) Figure 5 this has the added advantage of generating a wide variety of point data (e.g., degenerate cases that may not naturally occur in preloaded mesh models)
- Random point clusters i.e., to provide approximate performance metrics for the computational cost versus the number of points - Figure 7

We also emphasis failure cases, as shown in Figure 8, Figure 9, and Figure 10, which are caused when we do not include the additional spherical surface projection phase to reduce co-linear and coplanar issues. A point to note, is our approach reduces accuracy constraints (i.e., numerical sensitivity) for different curved surfaces by projecting the points onto a common spherical lattice. While we can use support mapping to extrapolate the surface vertices, incorrectly expanding the initial tetrahedron, due to numerical sensitivity and coplanar triangles, can produce mesh that engulfs the surface vertices but be concave, as shown in Figure 9.

The randomly generate point set tests of different sizes ranged from 500 to 10000 points. For each size, we generate 5 different datasets and average their run time. Figure 7 shows the run time details and shows our algorithm is $O(n \log n)$.

6 Discussion

A number of factors come into question, such as, computational speed and robustness. On the surface, a convex algorithm may ap-



Figure 6: Complex 3D Model - (a) Complex model of a gun (6174 points or 2058 faces), (b) wireframe view, and (c) generated convex shell (342 points or 114 faces).



Figure 7: Random 3D Cloud of Points - Approximate performance metrics measuring the time to generate a convex hull given different sets of randomly generated 3D points (i.e., from 500 to 10000 vertices).

pear elegant and straightforward but can be difficult to implement well. For example, for 3D point clouds, the solution can hit numerical issues for small hulls and computational bottlenecks for large numbers of points (10,000 or more vertices). While fixedpoint integer mathematics may help improve robustness, using realnumbers with limited accuracy (e.g., floats or doubles) makes the algorithm much faster but at a cost (e.g., stability and accuracy). We presented a novel generic method that works effectively with realnumbers that is able to deal with coplanar and collinear surfaces without any complex engineering enhancements or user interventions.

Our method works because a convex hull is defined by its vertices. The advantage of our method over other approaches is the surface vertices have already been found during the support mapping phase without overhead. In terms of complexity, it has the benefit of not having to do vertex culling when generating the convex surface. Additionally, when combined with sphere mapping, it reduces the need to perform swap operation to fix the convexity due to concave and other edges.

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Figure 5: *Minkowski Sum* - Summing two cube models using the Minkowski Sum to evaluate the Convex Hull implementation in a practical situation. Visually drawing each triangle normal allows us to easily identify the Convex Hull Mesh. (One cube is rotated while the second remains fixed for the Mikowski sum test cases shown in the illustration).





Figure 9: Failure Case 2 - Incorrectly removing and adding triangles iteratively may engulf all the surface vertices but produce a concave mesh (i.e., non-convex).



Figure 10: *Failure Case 3* - *Degenerate triangles (i.e., long thin triangles) can cause issues with the iterative evolution of the convex hull (e.g., incorrect normals). (a) long thin triangle, and (b) a degenerate triangle with a normal that is parallel to the surface.*

Figure 8: Failure Case 1 - Caution needs to be taken - while an implementation may seem correct on the surface - degenerate and troublesome triangles may be inserted within the convex hull (we can easily make out the troublesome inner triangles by seeing the normals).



Figure 11: Complex 3D Model - (a) Complex model of a cat (11862 points or 3954 faces), and (b) generated convex shell (162 points or 54 faces).

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